

A COMPLETE SET OF POSTULATES FOR THE THEORY OF ABSOLUTE CONTINUOUS MAGNITUDE*

BY

EDWARD V. HUNTINGTON

The following paper presents a complete set of postulates or primitive propositions from which the mathematical theory of absolute continuous magnitude can be deduced.

The fundamental concept involved is that of a *rule of combination* according to which any two objects of a given assemblage, in a definite order, determine a third object which itself belongs to the assemblage. (The same idea is the fundamental concept in the definition of a group.)

The rule of combination here considered is one upon which six fundamental restrictions are imposed, these requirements being expressed in the *postulates* † 1 to 6.

The object of the work which follows is to show that these six postulates form a *complete set*; that is, they are (I) *consistent*, (II) *sufficient*, (III) *independent* (or *irreducible*). By these three terms we mean: (I) there is at least one assemblage in which the chosen rule of combination satisfies all the six requirements; (II) there is essentially *only one* such assemblage possible; (III) none of the six postulates is a consequence of the other five.

This being shown (theorems I, II, III), the postulates 1–6 may be said to define a single assemblage, which we shall call the *system of absolute continuous magnitude*. The rule of combination may then be called *addition*.

From another point of view, the propositions 1–6 may be accepted as expressing in precise mathematical form the essential characteristics of magnitude in the popular sense of the word; from this point of view they may properly be called the *axioms* † of (absolute continuous) magnitude.

From either point of view, the propositions 1–6 form a complete logical basis for a deductive mathematical theory.

* Presented to the Society February 22, 1902. Received for publication March 5, 1902.

† Following the usual distinction, we use “postulate” to mean a proposition the acceptance of which is demanded or agreed upon as a basis for future reasoning, reserving “axiom” to mean “a self-evident proposition, requiring no formal demonstration to prove its truth, but received and accepted as soon as mentioned.”

The most recent work on this subject is a long and critical paper by O. HÖLDER (see bibliographical note *) to which the present article is closely related.

* *Bibliographical Note.*—The following references include papers in which the properties of magnitude are deduced from any system of postulates, whether or not the independence of the postulates is discussed.

G. VERONESE, *Il continuo rettilineo e l'assioma V. d'Archimede*, Atti della R. Accad. dei Lincei, Memoire d. cl. d. scienze fis., mat. e nat., ser. 4, vol. 6 (1889), pp. 603–624.

R. BETTAZZI, *Teoria delle grandezze*, 4° (1890), vii + 181 pp., Pisa. Awarded a prize by the R. Accad. dei Lincei. Reviewed in Darboux's Bulletin des sciences mathématiques, ser. 2, vol. 15 (1891), pp. 53–68.

O. STOLZ, *Ueber das Axiom des Archimedes*, Mathematische Annalen, vol. 39 (1891), pp. 107–112.

C. BURALI-FORTI, *Sulla teoria delle grandezze*, Formulaire de Mathématiques, vol. 1 (1895), pp. 28–57. Written in PEANO's mathematical language. See also Rivista di Matematica, vol. 3 (1893), pp. 76–101.

G. VERONESE, *Sul postulato della continuità*, Atti della R. Accad. dei Lincei, Rendiconti d. cl. d. Scienze fis., mat. e nat., ser. 5, vol. 6₂ (1897), pp. 161–168.

L. COUTURAT, *Sur la définition du continu*, Revue de metaphysique et de morale, vol. 8 (1900), pp. 157–168.

D. HILBERT, *Ueber den Zahlbegriff*, Jahresbericht der deutschen Mathematiker-Vereinigung, vol. 8 (1900), pp. 180–184.

E. V. HUNTINGTON, *Die Grund-Operationen an absoluten und complexen Grössen*, Braunschweig (1901), 63 pp.

O. STOLZ und J. A. GMEINER, *Theoretische Arithmetik*, Leipzig (1901). A second edition, now in publication, of STOLZ's *Allgemeine Arithmetik*, 1885, 1886.

O. HÖLDER, *Die Axiome der Quantität und die Lehre vom Mass*, Leipziger Berichte, Math.-Phys. Classe, vol. 53 (1901), pp. 1–64.

HÖLDER's main set of postulates, like the sets used in all the earlier papers, admits the idea of "greater and less" as a fundamental concept. In a footnote on p. 5, however, he works out, partially, a second set of postulates which does not involve this idea. The set of postulates adopted in the present paper is nearly the same as this second set of HÖLDER's, with the exception that DEDEKIND's postulate of continuity is here replaced by WEIERSTRASS's, and that two of HÖLDER's postulates, viz., [5] and the second part of [2], are here found to be deducible from the others. I am indebted to HÖLDER's paper for many valuable suggestions, especially for the proof of the commutative law (see 22, Case II, p. 271), which in all the earlier papers is admitted as a fundamental law.

The three following works, which I have not seen, may also contain matter bearing on this subject:

C. BURALI-FORTI, *Les propriétés formelles des opérations algébriques*, Turin (1900). See also Rivista di Matematica, vol. 6 (1899) pp. 141–177.

R. BETTAZZI, *Grandezza, quantità e numero*, Il Bollettino di Matematiche e di Scienze fisiche e naturali, Bologna, vol. 1 (1900), pp. 7–10, 70–73.

F. AMODEO, *Arithmetica particolare e generale*, Naples (1900).

A complete set of postulates for the theory of *positive integers with zero*, in which the fundamental concept is that of an element *successive* to a given element, is given by

G. PEANO, *Sul concetto di numero*, Rivista di Matematica, vol. 1 (1891), pp. 87–102, 256–267. See also Formulaire de Mathématiques, vol. 3 (1901), pp. 39–44. Essentially these postulates are reproduced in the footnote on p. 266.

A complete set of postulates for the theory of *all whole numbers* (positive, negative and zero) is given in two papers by

A. PADOA: 1) *Essai d'une théorie algébrique des nombres entiers*, presented to the International Congress of Philosophy, Paris, 1900, and reviewed in Rivista di Matematica, vol. 7

Fundamental concepts.

A *class* of objects is determined when any condition is given such that every object in the universe must either satisfy or not satisfy the condition. Every object which satisfies the condition is said to *belong to* the class.

(The condition might be such that no object would belong to the class; this case, however, we shall agree to exclude from the discussion—that is, we shall consider those classes only which contain at least one object.)

A class thus defined is usually called, in mathematical parlance, an *assemblage* (Menge, ensemble), every object which belongs to the class being called an *element* of the assemblage.*

A *rule of combination* in an assemblage is any rule or agreement by which, when any two elements (whether the same or different) are given, in a definite order, some object (which may or may not itself belong to the assemblage †) is uniquely determined.

If the first of the two given elements is denoted by a and the second by b , then the object which they determine is denoted by $a \circ b$ —read: “ a with b .”

The notation $x = y$ indicates that the two symbols x and y are used to represent the same object, while $x \neq y$ indicates that the objects represented by x and y are different.

In the proofs of theorems I and III illustrations are drawn from the ordinary artificial systems of real and complex numbers. In the rest of the paper, however, only the natural system of ordinal numbers ‡ is presupposed.

(1901), pp. 73–84; 2) *Un nouveau système irréductible des postulats pour l'algèbre*, presented to the International Congress of Mathematicians, Paris, 1900, reviewed in the Bulletin of the American Mathematical Society, 2d ser., vol. 7 (1900), p. 70. I have seen only the reviews of these papers. The fundamental concepts involved are the idea of an element *successive* to a given element, and the idea of an element *symmetrical* to a given element.

* H. WEBER, *Algebra*, vol. 2 (1899), p. 4.

† Compare postulate 1.

‡ The system of ordinal numbers is an assemblage characterized by the following fundamental properties, which PEANO (loc. cit.) has shown to be independent of one another:

1°) The assemblage contains a peculiar element, called the “first” ordinal number, and denoted by 1.

2°) Every element, m , of the assemblage determines uniquely an object m^+ , which itself belongs to the assemblage. This number m^+ is called the number next following the number m . (The number 1^+ is denoted by 2; 2^+ by 3, etc.)

3°) Whatever the number m may be, the number next following m is never 1.

4°) If the numbers next following two numbers m and n are the same, then m and n are the same.

5°) If S is any assemblage such that: (a) the number 1 belongs to S , and (b) the number next following m belongs to S whenever the number m belongs to S ; then S contains every ordinal number.

From these fundamental propositions follows the validity of the common method of “mathematical induction.” (See also 12.)

If any law is given by which every ordinal number k ($k = 1, 2, 3, \dots$) determines a definite object (a_k) the objects so determined are said to form an "infinite sequence," in notation $a_1, a_2, a_3, \dots, a_k, a_{k+1}, \dots$. It should be noticed, however, that a_p and a_q ($p \neq q$) do not necessarily represent different objects, since, according to the given law, it may be that an object is repeated many times in the sequence.

In the present paper we are concerned with the assemblages in which the rule of combination satisfies certain fundamental conditions, expressed in the six postulates of the following paragraph.

The postulates of magnitude.

1. *Whatever elements a and b may be ($a = b$ or $a \neq b$), $a \circ b$ is also an element of the assemblage.*

2. $a \circ b \neq a$.

3. $(a \circ b) \circ c = a \circ (b \circ c)$, whenever $a \circ b$, $b \circ c$, $(a \circ b) \circ c$ and $a \circ (b \circ c)$ belong to the assemblage.

4. *When $a \neq b$ at least one of the following conditions is satisfied: either 1°) there is an element x such that $a = b \circ x$, or 2°) there is an element y such that $a \circ y = b$.*

5.* *(Let the notation $a < b$ indicate that an element y exists such that $a \circ y = b$; and let $a \leq b$ indicate: $a < b$ or $a = b$.)*

If S is any infinite sequence of elements (a_k) , such that

$$a_k < a_{k+1}, \quad a_k < c \quad (k = 1, 2, 3, \dots)$$

(where c is some fixed element), then there is one and only one element A having the following two properties:

1°) $a_k \leq A$ whenever a_k belongs to S ;

2°) if y and A' are such that $y \circ A' = A$, then there is at least one element of S , say a_r , for which $A' < a_r$.

6. *Whatever element a may be, there are two elements x and y such that $x \circ y = a$; that is, in the notation explained in 5, there is an element x such that $x < a$.*

Consistency of the postulates.

THEOREM I.—*The postulates 1–6 are consistent with each other.*

To establish this theorem we have simply to exhibit some assemblage in which all the postulates are satisfied. Such an assemblage is the system of positive

* This postulate 5 is essentially the same as the principle employed by WEIERSTRASS, in his lectures, for the definition of an irrational number. But instead of saying, under 2°), "if $A' < A$," that is: "if a y exists such that $A' \circ y = A$," I have inverted the order of the A' and y . (Cf. 24.)

real numbers, when $a \circ b$ is defined as $a + b$. Another such assemblage is the system of positive real numbers greater than 1, when $a \circ b$ is defined as $a \times b$. (See also 36.)

Development of the theory in preparation for the proof of theorem II.

Remarks. We notice that the two systems mentioned in the proof of theorem I are *equivalent* (simply isomorphic); that is, the first can be put into one-to-one correspondence with the second in such a way that when a corresponds with a' and b with b' then $a \circ b$ will correspond with $a' \circ b'$. In theorem II we shall show that every assemblage which satisfies the postulates 1-6 will be equivalent with these. The propositions 7-35, which we proceed to deduce from the fundamental propositions 1-6, will prepare the way for the proof of theorem II.

Greater and less. Fundamental inequalities.

7. *Definition.* If with respect to two elements a, b an element x exists such that $a = b \circ x$, we shall write: $a > b$ (a greater than b) and $b < a$ (b less than a).

8. *Of the three relations $a = b$, $a > b$, $a < b$ of any two elements a and b*

1°) *not more than one can be true, and* (1, 2, 3)*

2°) *at least one must be true.* (4)

The proof for the first part is indirect.

9. *If $a > b$ and $b > c$ then $a > c$.*† (1, 3)

10. $a \circ b > a$. (1)

11. *If $b > b'$ then $a \circ b > a \circ b'$.* (1, 3)

For, from $b = b' \circ x$ it follows by 1 and 3 that $a \circ b = (a \circ b') \circ x$.

Multiples.

12. *Ordinal numbers.* On the basis of the fundamental properties mentioned in the introduction, we state here briefly such other properties of ordinal numbers as we shall have occasion to use.‡

a) *Definition.* The sum, $p + q$, of two numbers, p and q , is the number defined by the following recurrent formulæ:

$$p + 1 = p^+; \quad p + 2 = (p + 1) + 1; \quad p + 3 = (p + 2) + 1; \quad \dots; \\ p + (k + 1) = (p + k) + 1.$$

b) $p + q = q + p$. (Proof by mathematical induction.)

* With every theorem will be indicated the fundamental propositions from which it is deduced.

† From 8 and 9 we see that any assemblage which satisfies 1, 2, 3, 4 will possess *Größencharakter* in the sense defined by SCHÖNFLIES, *Jahresbericht der deutschen Mathematiker-Vereinigung*, vol. 8 (1900), *Bericht über die Mengenlehre*, p. 15.

‡ The most accessible proofs are in STOLZ and GMEINER, loc. cit., pp. 12-23.

c) *Definition.* The product, qp , of two numbers, p and q , is the number defined by the following recurrent formulæ:

$$1p = p; \quad 2p = 1p + p; \quad 3p = 2p + p; \quad \dots; \quad (k+1)p = kp + p.$$

d) $qp = pq$. (Proof by mathematical induction.)

e) If $p \neq q$, then there is either a number λ such that $p = q + \lambda$ or else a number μ such that $q = p + \mu$ (and not both). In the first case p is said to come later, in the second case earlier, than q in the sequence of ordinal numbers.

13. Multiples.

a) *Definition.* The m th multiple, ma , of any element a (m being any ordinal number) is the element defined by the following recurrent formulæ:

$$1a = a; \quad 2a = 1a \circ a; \quad 3a = 2a \circ a; \quad \dots; \quad (k+1)a = ka \circ a. \quad (1)$$

All the multiples of any element a clearly belong to the assemblage, by 1; but it should be noticed that there is nothing in postulate 1 to prevent some or all of these multiples from being the same. (See 13f.)

$$b) \quad pa \circ qa = (p + q)a. \quad (1, 3)$$

The proof by induction with respect to q involves 13a, 3, 12a.

$$c) \quad pa \circ qa = qa \circ pa. \quad (\text{By } 12b \text{ and } 13b.) \quad (1, 3)$$

$$d) \quad q(pa) = (qp)a. \quad (1, 3)$$

The proof by induction as to q involves 13a, 12c, 13b.

$$e) \quad q(pa) = p(qa) = qpa = pqa. \quad (\text{By } 12d \text{ and } 13d.) \quad (1, 3)$$

$$f) \quad pa = qa \text{ when and only when } p = q. \quad (1, 2, 3)$$

For, if $p = q$, then $pa = qa$, by identity. Conversely, if $pa = qa$, then $p = q$; for, suppose $p \neq q$, then by 12e either $p = q + \lambda$ or $q = p + \mu$, whence, by hypothesis and 13b, either $qa = qa \circ \lambda a$ or $pa = pa \circ \mu a$, relations each impossible, by 2.

$$g) \quad pa > qa \text{ when and only when } p \text{ comes later than } q. \text{ In particular, } pa \cong a. \quad (1, 2, 3)$$

For, if p comes later than q , then by 12e and 13b, $pa = qa \circ \lambda a$, whence, by 7, $pa > qa$. Conversely, if $pa > qa$, then p comes later than q ; for suppose p earlier than q , or else $p = q$, then by 12e, 13b, 13f we should have either $qa = pa \circ \mu a$ or else $pa = qa$, relations each impossible by 8, 1°).

$$h) \quad (b \circ pa) \circ (qa \circ c) = (b \circ qa) \circ (pa \circ c). \quad (\text{By } 13c \text{ and } 3.) \quad (1, 3)$$

Principle of Archimedes. Further inequalities.

14. *Principle of Archimedes.* Whatever elements a and b may be, there is a multiple of a , say ma , such that $ma > b$. (1, 2, 3, 4, 5)

The proof is indirect, as follows: Suppose the theorem to be false; then, by 8, $na \leq b$, or, by 10 and 9, $na < 2b$, for every multiple of a . Now by 13g the successive multiples of a form an infinite sequence of elements each of which is less than the succeeding one. From this fact and the supposition $na < 2b$ follows by 5 the existence of an element A having the following two properties:

1°) $na \leq A$ for every multiple of a ;

2°) if $y \circ A' = A$ there is at least one multiple of a , say $n'a$, such that $n'a > A'$.

From 1°) it follows that $a < A$, by 13g and 9; therefore there is by 7 an element A' such that $a \circ A' = A$. Hence, by 2°), we have $n'a > A'$ for at least one multiple of a . From this relation follows, by 11, $a \circ n'a > a \circ A'$, that is, $a \circ n'a > A$, whence, by 13c and 13a, $(n' + 1)a > A$. This last result, however, is in contradiction with 1°). Therefore the theorem must be true.

15. If $a < b$ there are two successive multiples of a , say pa and $(p + 1)a$, such that

$$pa < b \leq (p + 1)a. \quad (1, 2, 3, 4, 5)$$

For, there is one multiple of a , viz., $1a$, which is, by hypothesis, less than b . Hence, if it were true that $(p + 1)a < b$ whenever $pa < b$, it would follow by mathematical induction that $ma < b$ for every multiple of a , a result impossible by 14.

$$16. a \circ b > b. \quad (1, 2, 3, 4, 5)$$

When $a \equiv b$ the theorem is clearly true, by 10 and 9.

When $a < b$ we have, by 15, $pa < b \leq (p + 1)a$. From $pa < b$ follows, by 11, $a \circ pa < a \circ b$, or, by 13c and 13a $(p + 1)a < a \circ b$. But $b \leq (p + 1)a$; hence, by 9, $b < a \circ b$.

$$17. \text{ If } a > a' \text{ then } a \circ b > a' \circ b. \quad (\text{By } 7, 1, 3, 16, 11.) \quad (1, 2, 3, 4, 5)$$

$$18. \text{ If } a > a' \text{ and } b > b' \text{ then } a \circ b > a' \circ b'. \quad (1, 2, 3, 4, 5)$$

(By 11, 17, 9.)

19. Let $a \circ b = a' \circ b'$; then from $b = b'$ follows $a = a'$, and from $b > b'$ follows $a < a'$. (1, 2, 3, 4, 5)

For the other suppositions (8) lead to contradictions with 17, 11 or 18.

$$20. \text{ If } a > a' \text{ then } ma > ma'. \quad (1, 2, 3, 4, 5)$$

The proof by induction involves 18 and 13a.

$$21. \text{ If } ma = ma' \text{ then } a = a'; \text{ if } ma > ma' \text{ then } a > a'. \quad (1, 2, 3, 4, 5)$$

For the other suppositions (8) lead to contradictions with 20.

The commutative law.

22. Commutative law.

$$a \circ b = b \circ a. \quad (1, 2, 3, 4, 5)$$

It is clear that in any assemblage which satisfies 1-5, the postulate 6 must be either true or false. By considering both cases, we shall prove the commutative law from 1, 2, 3, 4, 5 without using 6.

Case I. Let postulate 6 be false; then the given assemblage contains a minimum element E , such that whatever element a may be $a \cong E$. Then every element of the assemblage is some multiple of E .

For, suppose there were an element c not a multiple of E ; then, since $E < c$, we have, by 15, $pE < c < (p+1)E$. Hence, by 7, $pE \circ y = c$ and $c \circ z = (p+1)E$, whence $(pE \circ y) \circ z = (p+1)E$, or, by 3 and 13a, $pE \circ (y \circ z) = pE \circ E$. But from this follows, by 19, $y \circ z = E$, or, by 7, $y < E$, which is impossible in this case.

Since every element is thus of the form mE , the commutative law follows at once from 13c.

*Case II.** Let postulate 6 be true; then whatever element a may be, there are two elements x and y such that $x \circ y = a$. The proof for this case is indirect.

Suppose $a \circ b < b \circ a$; then, by 7, $(a \circ b) \circ y = b \circ a$.

By 6 take $y' \circ y'' = y$, and x less than the least of the four elements y' , y'' , a , b . From $x < y'$ and $x < y''$ follows, by 18, $x \circ x < y' \circ y''$, that is,

$$x \circ x < y.$$

Since $x < a$ and $x < b$, we can take p and q by 15 such that

$$px < a \cong (p+1)x$$

and

$$qx < b \cong (q+1)x.$$

From the first parts of these inequalities, we have, by 18, $px \circ qx < a \circ b$, whence, by 17, $(px \circ qx) \circ y < (a \circ b) \circ y$, that is:

$$A) \quad (px \circ qx) \circ y < b \circ a.$$

From the second parts of the same inequalities, reversing the order, we have, by 18 and 13a, $b \circ a \cong (qx \circ x) \circ (px \circ x)$, or by 13h

$$b \circ a \cong (qx \circ px) \circ (x \circ x),$$

whence by 11 and 9

$$B) \quad b \circ a < (qx \circ px) \circ y.$$

These two results, A) and B), are, however, inconsistent, by 8; for by 13c

$$px \circ qx = qx \circ px;$$

therefore the supposition from which they were deduced is false.

In a similar way we can show that the supposition $a \circ b > b \circ a$ is also false. Hence by 8 the third possibility, viz., $a \circ b = b \circ a$, must be true.

* Cf. HOLDER, loc. cit., p. 13.

$$23. (b \circ x) \circ (y \circ c) = (b \circ y) \circ (x \circ c). \quad (1, 2, 3, 4, 5)$$

The proof is the same as the proof of 13*h*, when 22 is used in place of 13*c*.

Limit of an increasing sequence.

24. If S is any infinite sequence of elements (a_k) , each of which is less than the succeeding one, and also less than some fixed element c :

$$a_k < a_{k+1}, \quad a_k < c \quad (k=1, 2, 3, \dots),$$

then there is one and only one element A having the following two properties:

1°) $a_k \equiv A$ whenever a_k belongs to S ;

2°) if $A' < A$ there is at least one element of S , say a_r , for which $A' < a_r$.

The element A thus determined is called the limit of the infinite sequence S and is denoted by $\lim (a_k)$. (1, 2, 3, 4, 5)

Proof. Whenever $A' < A$ there is an element y such that $y \circ A' = A$; for by 7 there is an element y such that $A' \circ y = A$, and by 22

$$A' \circ y = y \circ A'.$$

Therefore by 5 there is at least one element A which has the properties 1°) and 2°). Further two elements A, B each with both these properties are the same. For suppose $B < A$. Then by 2°) as to A there is an element of S which is greater than B , while by 1°) as to B no element of S is greater than B .

$$25. \text{ If } \lim (a_k) = A \text{ and } \lim (b_k) = B \text{ then } \lim (a_k \circ b_k) = A \circ B. \quad (1, 2, 3, 4, 5)$$

We consider here two cases, as under 22.

Case I. Let postulate 6 be false. Then there is a minimum element E such that, whatever element a may be, $a \equiv E$; whence, as we saw in 22, Case I, every element is some multiple of E . In this case, by 13*g*, there cannot be an infinite number of elements less than a fixed element $c = mE$, and hence the conditions in the hypothesis cannot be fulfilled.

Case II. Let postulate 6 be true. Then whatever element a may be, there are two elements x and y such that $x \circ y = a$. The first condition 1°) for the truth of the theorem is in this case clearly satisfied; for by hypothesis and 24, 1°) $a_k \equiv A$ and $b_k \equiv B$; hence, by 18, $a_k \circ b_k \equiv A \circ B$ for every index k .

To show that the second condition, 2°), is also satisfied, let C' be any element $< A \circ B$. By 7, $C' \circ y = A \circ B$; by 6 take $y' \circ y'' = y$ and x less than the least of the elements y', y'', A, B ; and by 7 and 22 take A' and B' such that $A' \circ x = A$, $B' \circ x = B$. Hence $(A' \circ x) \circ (B' \circ x) = A \circ B$, that is, by 23, $(A' \circ B') \circ (x \circ x) = C' \circ (y' \circ y'')$, while, by 18, $x \circ x < y' \circ y''$. Therefore by 19, $A' \circ B' > C'$.

But by 7, $A' < A$ and $B' < B$; therefore by 24, 2°) there is an element

$a_r > A'$ and an element $b_t > B'$. Let r be an index that comes later than s and t ; then by 9, $a_r > A'$ and $b_r > B'$, whence, by 18, $a_r \circ b_r > A' \circ B'$.

Therefore, by 9, $a_r \circ b_r > C'$, and the theorem is established.

26. If $\lim (a_k) = A$ then $\lim (ma_k) = mA$. (1, 2, 3, 4, 5)

The inductive proof involves 13a and 25.

Submultiples.

27. Lemma. Given a and m there is an element y such that $my < a$.

(1, 2, 3, 4, 5, 6)

If the theorem is true when $m = n$, we shall have, by 7, $nz \circ z' = a$, and by 6 we can take y less than z and z' . Then, by 20, $ny < nz$ and, by 18, $ny \circ y < nz \circ z'$, whence, by 13a, $(n+1)y < a$; that is, the theorem will be true when $m = n+1$. But it is clearly true when $m = 1$, by 6; hence it will be true for every value of m .

28. Given any element a and any ordinal number m there is one and only one element x such that $mx = a$.

The element x thus determined is called the m th submultiple of a and is denoted by a/m . Thus $m(a/m) = a$. (1, 2, 3, 4, 5, 6)

The proof in outline is this: we construct an infinite sequence of elements x_1, x_2, x_3, \dots , which has by 24 a limit x ; we then prove that the sequence mx_1, mx_2, mx_3, \dots has the limit a , whence it follows that x is the element required in the theorem.

a) The first element of the sequence is found as follows:

Take y_1 by 27 so that $my_1 < a$, and then p_1 by 15 so that

$$p_1(my_1) < a \leq (p_1 + 1)(my_1);$$

hence, by 13e and 13a, $m(p_1 y_1) < a \leq m(p_1 y_1) \circ my_1$.

Then for the first element of the sequence we take $x_1 = p_1 y_1$, and by 7 we have $mx_1 \circ e_1 = a$.

b) When any element, as the k th, has been found, the next following element—the $(k+1)$ th—is obtained as follows:

We have

$$(1) \quad mx_k \circ e_k = a.$$

By 27 take y_{k+1} so that

$$(2) \quad (k+1)my_{k+1} < e_k,$$

whence, by 13g and 16,

$$(3) \quad my_{k+1} < e_k < a.$$

Then take p_{k+1} by 15 so that $p_{k+1}(my_{k+1}) < a \leq (p_{k+1} + 1)(my_{k+1})$, whence by 13e and 13a

$$(4) \quad m(p_{k+1} y_{k+1}) < a \leq m(p_{k+1} y_{k+1}) \circ my_{k+1}.$$

For the $(k + 1)$ th element we then take $x_{k+1} = p_{k+1}y_{k+1}$, and by 7 we have

$$(5) \quad mx_{k+1} \circ e_{k+1} = a.$$

c) In the sequence thus determined we have from (4) and (5)

$$m(x_{k+1}) \circ e_{k+1} \leq m(x_{k+1}) \circ my_{k+1};$$

hence by 8 and 11

$$(6) \quad e_{k+1} \leq my_{k+1}.$$

From (6) and (3) we have, by 9, $e_{k+1} < e_k$, while from (1) and (5)

$$mx_{k+1} \circ e_{k+1} = mx_k \circ e_k = a.$$

Hence by 19 and 10

$$(7) \quad mx_k < mx_{k+1} < a,$$

and by 21, 13g and 9

$$(8) \quad x_k < x_{k+1} < a.$$

Further, if c is any element, there is an index r such that $e_r < c$. For, if $c \leq e_k$ for every index k , then $c \leq e_{k+1}$, or by 20, (6), (2), and (3)

$$(k + 1)c \leq (k + 1)e_{k+1} \leq (k + 1)my_{k+1} < e_k < a$$

for every ordinal number $k + 1$, which is impossible by 14.

From this it follows that if $a' < a$ then there is an index r such that

$$(9) \quad a' < mx_r.$$

For, let $a' \circ z = a$ by 7 and take $e_r < z$. Then by (1) $a' \circ z = mx_r \circ e_r$, whence, by 19, $a' < mx_r$.

d) The theorem is now readily established; for, from (8) it follows by 24 that the sequence x_1, x_2, x_3, \dots has a limit: $\lim(x_k) = x$; hence by 25 $\lim(mx_k) = mx$. But from (7) and (9) we have by 24 $\lim(mx_k) = a$. Therefore $mx = a$. That no other element, x' , can have this property follow at once from 21.

29. If p comes later than q in the sequence of ordinal numbers (12e), then $a/p < a/q$. In particular, $a/p \leq a$. (1, 2, 3, 4, 5, 6)

For, suppose $a/p \geq a/q$; then, by 13g and 20, $p(a/p) > q(a/p) \geq q(a/q)$, whence, by 28, $a > a$, which is impossible.

30. Whatever elements a and b may be, there is a submultiple of a , say a/m , such that $a/m < b$. (By 8, 20, 28, 14.) (1, 2, 3, 4, 5, 6)

Rational fractions.

31. Definition. Any multiple (13a) of any submultiple (28) (or any submultiple of any multiple) of an element a , is called a rational fraction of a ; in notation pa/q . (1, 2, 3, 4, 5, 6)

This (ambiguous) notation is allowable, for by 13e and 28 we have $q[p(a/q)] = p[q(a/q)] = pa$, whence, by 28,

$$p(a/q) = (pa)/q.$$

32. The following theorems are deduced from 31 by the aid of 12, 13, 21 and 28. (1, 2, 3, 4, 5, 6)

a) $mpa/mq = pa/q$.

For, $mq(mpa/mq) = mpa = mq(pa/q)$, by 13e and 28. Hence follows the theorem, by 21.

b) $pa/q \circ p'a/q' = (q'p + qp')a/qq'$.

For, $pa/q = q'p(a/qq')$ and $p'a/q' = qp'(a/qq')$, by a) and 12d. Hence follows the theorem, by 13b.

c) $pa/q = p'a/q'$ when and only when $q'p = qp'$.

d) $pa/q > p'a/q'$ when and only when $q'p$ comes later than qp' .

These two theorems follow by 13f and 13g from the equations given under b).

e) If $pa/q < p'a/q'$ there is a rational fraction of a , say ma/n , such that $pa/q \circ ma/n = p'a/q'$.

Since by d) $q'p$ comes earlier than qp' , there is by 12e an ordinal number μ such that $q'p + \mu = qp'$, and if we take $ma/n = \mu a/qq'$, we shall satisfy the theorem. For by a), 12d and 13b

$$pa/q \circ ma/n = q'p(a/qq') \circ \mu(a/qq') = (q'p + \mu)a/qq' = qp'a/qq' = p'a/q'.$$

33. Given c , and $a < b$, we can find mc/n such that $a < mc/n < b$.

(1, 2, 3, 4, 5, 6)

Let $a \circ y = b$, by 7. By 30 and 16 take $c/n < y < b$, and by 15 take m so that $m(c/n) < b \equiv (m+1)(c/n)$. Then $a < mc/n$. For, if $a \equiv mc/n$, then, by 18, $a \circ y > mc/n \circ c/n$, or, by 13a, $b > (m+1)c/n$, which is impossible.

34. Given $mc/n < a \circ b$, we can find $pc/q < a$ and $p'c/q' < b$ such that $pc/q \circ p'c/q' = mc/n$. (1, 2, 3, 4, 5, 6)

To fix our ideas, let $a \equiv b$.

If $mc/n \equiv b$, take $c/q < a$ and $< mc/n$, by 30, and by 32e take $p'c/q'$ so that $c/q \circ p'c/q' = mc/n$. Then by 16 and 9, $p'c/q' < b$.

If $b < mc/n$, we have, by 7 and 22, $y \circ b = mc/n < a \circ b$, whence, by 17, $y < a$. By 33 and 9 take pc/q so that $y < pc/q < a \equiv b < mc/n$ and by 32e take $p'c/q'$ so that $pc/q \circ p'c/q' = mc/n = y \circ b$, whence, by 19, $p'c/q' < b$.

Upper limit of an assemblage of rational fractions.

35. If U is any assemblage of rational fractions of a such that every element of U is less than some fixed element c , then there is one and only one element A having the following two properties:

1° every element of U is $\equiv A$;

2° if $a' < A$, there is at least one element of U which is $> a'$.

The element A thus determined is called the upper limit of the assemblage U , and is denoted by $\overline{L}(U)$. (1, 2, 3, 4, 5, 6)

Proof:

Case I.—Suppose the assemblage U contains a maximum element; then this maximum element will clearly be the element A required in the theorem.

Case II.—Suppose on the other hand that U does not contain a maximum element. Then when any element of U is given we can find another element of U greater than the given one; whence it follows that U is an infinite assemblage.

To establish the theorem in this case we make use of a device due to G. CANTOR by which the assemblage of all the rational fractions of a can be arranged in an infinite sequence, R , as follows:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	...
$\frac{1}{1}a$;	$\frac{1}{2}a$,	$\frac{2}{1}a$;	$\frac{1}{3}a$,	$\frac{2}{2}a$,	$\frac{3}{1}a$;	$\frac{1}{4}a$,	$\frac{2}{3}a$,	$\frac{3}{2}a$,	$\frac{4}{1}a$;	$\frac{1}{5}a$,	$\frac{2}{4}a$,	$\frac{3}{3}a$,	$\frac{4}{2}a$,	$\frac{5}{1}a$;	...

Taking this complete sequence R we first cast out every element which does not belong to U ; what is left is a sequence U' containing all the elements of U and no others. U' will be an infinite sequence, since U is an infinite assemblage, but its elements will not, in general, be arranged in order of magnitude.

Out of the sequence U' we then select the elements x_1, x_2, x_3, \dots as follows: x_1 is the first element of U' ; x_2 is the first element beyond x_1 which is greater than x_1 ; x_3 is the first element beyond x_2 which is greater than x_2 ; and so on; in general, x_{k+1} is the first element of U' which is $> x_k$. (When any element x_k of this sequence is given, the next element, x_{k+1} , can surely be found, since otherwise x_k would be a maximum element, which contradicts the hypothesis.)

The elements x_1, x_2, x_3, \dots thus determined form an infinite sequence, S , in which every element is less than the succeeding one, while all (being elements of U) are less than the fixed element c . Hence by 24 the sequence S determines a definite element A as its limit: $\lim(x_k) = A$.

Now first, every element of U will be $\leq A$. For, suppose $pa/q > A$ were an element of U , and hence occupied a place in the sequence U' . Then either pa/q would itself belong to S , or else some earlier element of U' , which is $\leq pa/q$, would belong to S ; in either case we should have an element of S greater than A , which is impossible, by 24, 1°).

And secondly, if $A' < A$, there is an element of U which is $> A'$. For, by 24, 2°), there is an element of S which is $> A'$, and this element belongs also to U .

Therefore the element A satisfies both the conditions in the theorem.

Further, no other element can satisfy both these conditions. (Proof as in the latter part of 24.) Hence the theorem is true in all cases.

Sufficiency of the postulates to define an assemblage.

THEOREM II.—*Any two assemblages M and M' which satisfy the postulates 1–6 are equivalent; that is, they can be brought into one-to-one correspondence in such a way that $a \circ b$ will correspond with $a' \circ b'$ whenever a and b in M correspond with a' and b' in M' respectively.*

First, take any element E of M and any element E' of M' , to serve as "unit elements."

a) *To every element a of M we can assign a definite element a' of M' , as follows:*

Let U be the assemblage of all rational fractions of E which are less than a ,* and let U' be the assemblage comprising the same rational fractions of E' . (That is, pE'/q is to belong to U' when, and only when, pE/q belongs to U .) Now by 9 every element of U is less than some fixed rational fraction mE/n (where $a < mE/n$). Therefore by 32d every element of U' will be less than mE'/n . Hence by 35 the assemblage U' will have an upper limit, a' ; this element a' we then assign to a . We have then $\overline{L}(U) = a$ and $\overline{L}(U') = a'$.

b) *Similarly to every element of M' we assign a definite element of M .*

c) *Further, if a' is the element assigned to a , then a will be the element assigned to a' .*

To prove this it is sufficient to show that every rational fraction of E' which is less than a' belongs to U' . Thus, let pE'/q be any such fraction. Then by 24, 2°), since $\overline{L}(U') = a'$, there is an element of U' , say $\lambda E'/\mu$, for which $pE'/q < \lambda E'/\mu$, whence, by 32d, $pE/q < \lambda E/\mu$. But $\lambda E/\mu < a$, since otherwise $\lambda E'/\mu$ could not belong to U' ; hence, by 9, $pE/q < a$, and therefore pE'/q belongs to U' . Thus a strictly one-to-one correspondence is established between M and M' .

d) To prove the second part of the theorem, let U, V and W be the assemblages comprising all the rational fractions E which are less than a, b and $a \circ b$ respectively.

Then, if x belongs to U and y to V , $x \circ y$ will belong to W , by 32b and 18. And if z belongs to W we can find an x of U and a y of V such that $x \circ y = z$, by 34.

Thus the assemblage W is composed wholly of elements of the form $x \circ y$.

Similarly, if U', V' and W' are the assemblages comprising all rational fractions of E' which are less than $a', b', a' \circ b'$ respectively, then W' will be composed wholly of elements of the form $x' \circ y'$, where x' and y' belong respectively to U' and V' .

Now by hypothesis U and U' (and also V and V') comprise the same rational fractions of E and E' respectively. Hence, by 32b, W and W' will also comprise the same rational fractions of E and E' respectively; that is, $a' \circ b'$ will correspond with $a \circ b$.

* Cf. DEDEKIND'S idea of a cut (Schnitt) in his *Stetigkeit und irrationale Zahlen*.

36. *From Theorems I and II we may say that the postulates 1-6 define essentially a single assemblage. This assemblage we call the system of absolute continuous magnitude, and the rule of combination addition.*

For example, the two systems of artificial numbers mentioned in the proof of Theorem I are systems of magnitude in the sense here defined.

Another example (if we admit the usual geometric axioms) is the system of distances laid off along a line in one direction from a fixed point, $a \circ b$ being defined as the combined length of a and b , or, equally well, as the length of the hypotenuse of the right triangle whose legs are a and b .*

Independence of the postulates.

THEOREM III.—*The postulates 1-6 are independent; that is, no one of them is a consequence of the other five.*

The independence of any one of the postulates will be proved if we can find any assemblage which satisfies all the other postulates, but not the one in question.† We give below an assemblage of the required sort for each of the six postulates, thus establishing the theorem.

M_1) Let M_1 be the assemblage of all positive real numbers less than 10: $0 < x < 10$; and let $a \circ b$ be defined as $a + b$. Then M_1 satisfies all the other postulates, but not 1.

Postulate 1 is clearly not satisfied, since, for example, $5 \circ 8 = 13$ is not an element of the assemblage. The other postulates all hold.

M_2) Let M_2 be the assemblage of all real numbers, while $a \circ b = a + b$. Then M_2 satisfies all the other postulates, but not 2.

Postulate 2 does not hold, for $a \circ 0 = a$. Postulates 1, 3, 4 and 6 clearly hold. In 5 notice that whatever elements a and b may be there is always an element y such that $a \circ y = b$, so that by the definition of the symbol $<$, we shall have $a < b$ for all values of a and b . Hence S may be any infinite sequence of elements, and any element will answer for c ; then 5 is satisfied, since any element may be taken as A .

M_3) Let M_3 be the assemblage of all real numbers, while $a \circ b$ is defined as follows: when $a \neq b$, $a \circ b = (a + b)/2$; when $a = b$, $a \circ a = a + 1$.

Then M_3 satisfies all the other postulates, but not 3.

The third postulate does not hold, since, for example, $(2 \circ 6) \circ 10 = 7$, while $2 \circ (6 \circ 10) = 5$. Postulate 1 is clearly satisfied; 2 also holds, since $(a \circ b)/2 \neq a$ when $a \neq b$. In 4 we can take $y = 2b - a$ and in 6 we can take $x = a - 1 = y$; hence 4 and 6 are satisfied. In 5 notice that $a < b$ when-

* This last illustration is used by H. WEBER, loc. cit., vol. 1, p. 9.

† This method was employed by PEANO (loc. cit.) and by HILBERT in his *Grundlagen der Geometrie*, Leipzig, 1899.

ever $a \neq b$, since whenever $a \neq b$ there is an element y , viz., $y = 2b - a$, for which $a \circ y = b$. Hence S may be any infinite sequence in which each element is different from the one next following; and c may be any element not belonging to S . We can then take as A the element c , and 5 is satisfied.

M_4) Let M_4 be the assemblage of all the couples of the form (α, β) where α and β are positive real numbers; and let the rule of combination be the following:

$$(\alpha_1, \beta_1) \circ (\alpha_2, \beta_2) = (\alpha_1 + \alpha_2, \beta_1 + \beta_2).$$

Then M_4 satisfies all the other postulates, but not 4.

(The couples (α, β) may be represented as the points in the first quadrant of the complex plane, excluding the axes.) To show that 4 does not hold, consider, for example, $a = (2, 5)$, $b = (2, 4)$. Postulates 1, 2, 3 and 6 clearly hold. In 5 notice that when $(\alpha_1, \beta_1) < (\alpha_2, \beta_2)$ then $\alpha_1 < \alpha_2$ and $\beta_1 < \beta_2$, according to the definition of the symbol $<$. Hence, under the conditions in 5, the α 's of the couples which belong to S have an upper limit ξ , and the β 's an upper limit, η . If now we take $A = (\xi, \eta)$ we see that 5 is satisfied.

M_5) Let M_5 be the assemblage of positive rational numbers, while $a \circ b = a + b$. Then M_5 satisfies all the other postulates, but not 5.

Postulate 5 is not satisfied; for, consider any sequence of increasing rational fractions used to define any irrational number, as $\sqrt{2}$. All the other postulates clearly hold.

M_6) Let M_6 be the assemblage of positive integers, while $a \circ b = a + b$. Then M_6 satisfies all the other postulates, but not 6.

Postulate 6 clearly does not hold, since M_6 contains the minimum element 1. Postulate 5 is satisfied; for no infinite sequence S which satisfies the conditions stated can occur. All the other postulates clearly hold.

Two unsolved problems.

1. Suppose that in 5, 2°) the condition: " $y \circ A' = A$ " be replaced by: " $A' \circ y = A$." Can 14 and 22 be deduced from the postulates in this form, or must a new postulate like this: * "Given a and b , there is an x such that $a \circ b = b \circ x$ "—be added? In the latter case, how is the independence of the new postulate to be proved?

2. Suppose that in 5 the "sequence S " be replaced by: "any assemblage of elements all of which are $< c$." Is postulate 4 then deducible or still independent?

HARVARD UNIVERSITY,
CAMBRIDGE, MASS.

* Cf. HÖLDER, loc. cit., p. 5, footnote, axiom [5].